2 The linear transport equation

2.1 Derivation

Let u denote the density (i.e., mass per volume unit) of some substance and I assume that the geometry of the problem is well approximated by a line (think about some pollutant in a river or cars on a road) with the coordinate x. Therefore I deal with a spatially one-dimensional problem, and my density at the point x at time t is given by $u(t,x)$. I assume additionally that the total amount of u stays constant (no sources or sinks) and calculate the change of the substance in some arbitrary interval (x_1,x_2). First, at time t I will have that the total amount of u in (x_1,x_2) is

$$\int_{x_1}^{x_2} u(t,x)\mathrm{d}x,$$

and its change at time t is, assuming that u is smooth enough,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1}^{x_2} u(t,x)\mathrm{d}x=\int_{x_1}^{x_2} u_t(t,x)\mathrm{d}x.$$

At the same time the same change is given by

$$q(t,x_1)-q(t,x_2),$$

where $q(t,x)$ is the *flux* of the same substance, i.e., the change of this substance per time unit at the point x. Basically, $q(t,x_1)$ tells me how much substance enters the interval (x_1,x_2) at time t at the left boundary and $q(t,x_2)$ tells me how much substance leaves (x_1,x_2) at the right boundary, hence the choice of the signs. By the *conservation law*

$$\int_{x_1}^{x_2} u_t(t,x)\mathrm{d}x=q(t,x_1)-q(t,x_2)=-\int_{x_1}^{x_2} q_x(t,x)\mathrm{d}x,$$

where the last equality holds by the fundamental theorem of calculus. Finally, since my interval is arbitrary, I conclude that my substance must satisfy the equation

$$u_t+q_x=0.$$

To arrive to the final equation I must decide on the connection between u and q. In the simplest case

$$q(t,x)=cu(t,x)$$

for some constant c I end up with the linear one-dimensional transport equation

$$u_t+cu_x=0. \tag{2.1}$$

Keeping in mind the physical interpretation of (2.1), I will need also some *initial condition*

$$u(0,x)=g(x), \tag{2.2}$$

for some given function g (initial density) and, if my system is not spatially infinite, some *boundary* conditions (see below). At this point I assume that it is a good approximation to let $x\in(-\infty,\infty)$ and hence no need for boundary conditions. My next goal is to show that I can always find a solution to the problem (2.1)–(2.2) and, even more importantly, that this solution is physically relevant (the problem is *well posed*).

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2.2 Solving the one-dimensional transport equation. Traveling waves

The key point to "guess" a solution to (2.1) is to realize that (2.1) actually gives a lot of geometric information about the possible solution u, which in the considered case of two variables (t, x) is a surface in three dimensional space. Note that the gradient of u is given by

$$\operatorname{grad} u = \nabla u = (u_t, u_x),$$

and hence equation (2.1) can be written equivalently as

$$\boldsymbol{\gamma} \cdot \nabla u = 0, \quad \boldsymbol{\gamma} = (1, c).$$

Here dot denotes the usual dot product between two vectors. We know that if the dot product is zero hence these two vectors are orthogonal:

$$\boldsymbol{\gamma} \perp \nabla u.$$

At the same time, the gradient of the function points in the direction of the fastest increase and, hence, perpendicular to the level sets $u(t, x) = \text{const}$, which are curves on the plane (t, x). Finally, I conclude that $\boldsymbol{\gamma}$ is parallel to the level sets $u(t, x) = \text{const}$, therefore the level sets are the straight lines with the direction vector $\boldsymbol{\gamma}$. I can rephrase the last sentence as the sought function u is constant along any straight line with the direction vector $\boldsymbol{\gamma}$. All such straight lines can be written as $x = ct + \xi$, where ξ is some constant. Now I claim that this geometric interpretation I discussed is enough to find the solution to our problem. Namely, since I know that u is constant along $x = ct + \xi$ then the value of u at an arbitrary point (t^*, x^*) is the same along the line $x = ct + x^* - ct^*$. At the initial time $t = 0$ I will get the unique point $x = x^* - ct^*$, and the value of $u(t^*, x^*)$ must be equal to $g(x) = g(x^* - ct^*)$. Therefore, I found, dropping the asterisks, that the unique solution to the problem (2.1)–(2.2) is

$$u(t, x) = g(x - ct).$$

I am aware that for some people the geometric arguments I presented are not very convincing (but please see the figures below) therefore I will present a totally algebraic argument, which also will allow me to find a *general* solution to the linear transport equation. The key idea is that I should try to consider (2.1) in the new coordinates (which I "guessed" from the geometry of the equation)

$$\tau = t, \quad \xi = x - ct,$$

(note that in many cases t is used instead of τ, but I would like to emphasize that my τ and ξ are *new* coordinates and do not confuse them with *old* t and x). This change of variables is clearly invertible.

I have

$$u(t, x) = u(\tau, ct + \xi) = v(\tau, \xi) = v(t, x - ct).$$

Now I can use the chain rule. To wit,

$$\frac{\partial u}{\partial t}(t, x) = \frac{\partial v}{\partial t}(t, x - ct) = \frac{\partial v}{\partial \tau}(\tau, \xi)\frac{\partial \tau}{\partial t} + \frac{\partial v}{\partial \xi}(\tau, \xi)\frac{\partial \xi}{\partial t} = \frac{\partial v}{\partial \tau}(\tau, \xi) - c\frac{\partial v}{\partial \xi}(\tau, \xi).$$

Similarly,

$$\frac{\partial u}{\partial x}(t, x) = \frac{\partial v}{\partial \xi}(\tau, \xi).$$

After plugging the found expressions into (2.1) I get

$$v_\tau - cv_\xi + cv_\xi = 0 \implies v_\tau = 0,$$

which we already solved in Lecture 1. Therefore,

$$v(\tau, \xi) = F(\xi),$$

and hence, returning to the original coordinates,

$$u(t, x) = F(x - ct),$$

where F is an arbitrary $\mathcal{C}^{(1)}$ function (recall that I am looking for a classical solution). This is the *general solution* to equation (2.1).

To use the initial condition (2.2) I take $t = 0$ and get, as expected, that $F(x) = g(x)$ hence $F = g$.

Summarizing,

Theorem 2.1. *Problem* (2.1) *has the general solution*

$$u(t, x) = F(x - ct)$$

for an arbitrary $F \in \mathcal{C}^{(1)}(\mathbf{R}; \mathbf{R})$ *function.*

The initial value problem (2.1)*,* (2.2) *with* $g \in \mathcal{C}^{(1)}$ *has a unique classical solution*

$$u(t, x) = g(x - ct).$$

Theorem 2.1 is an *existence and uniqueness* theorem for the initial value problem for the linear one dimensional transport equation.

The straight lines $x = ct + \xi$ are very important and called the *characteristics*. Hence we now know that the solutions to the transport equation are constant along the characteristics.

What is the geometric meaning of the function of the form $(x, t) \mapsto g(x - ct)$? Take, e.g., $c > 0$ then for fixed time moments $t_0 = 0, t_1 > t_0, t_2 > t_1, \ldots$ I will get the same graph of g only shifted by $0, ct_1, ct_2, \ldots$ units to the right. What I observe is a *traveling wave* moving from left to right with the speed c. If $c < 0$ then my wave will travel with the speed $|c|$ from right to left. This geometric picture should explain the title *transport equation*, since, according to the analysis above, the equation describes the transportation of the substance u from the point $(0, \xi)$ to some other point (t, x) along the characteristic $x = ct + \xi$.

Example 2.2. Here is a simple graphical illustration of solutions to the problem

$$u_t + u_x = 0,$$

with the initial condition

$$u(0, x) = e^{-x^2}.$$

We have that the solution is $u(t, x) = e^{-(x-t)^2}$, whose graphs at different time moments are given in Fig. 1.

In Fig. 2 one can see the same solution in the form of full three dimensional surface (the left panel) and a contour plot (the right panel), on which the level sets are represented.

Finally, I can put everything together in the same figure (Fig. 3), where you can see solutions at different time moments (bold curves), the three dimensional surface, together with the characteristics (bold dashed lines).

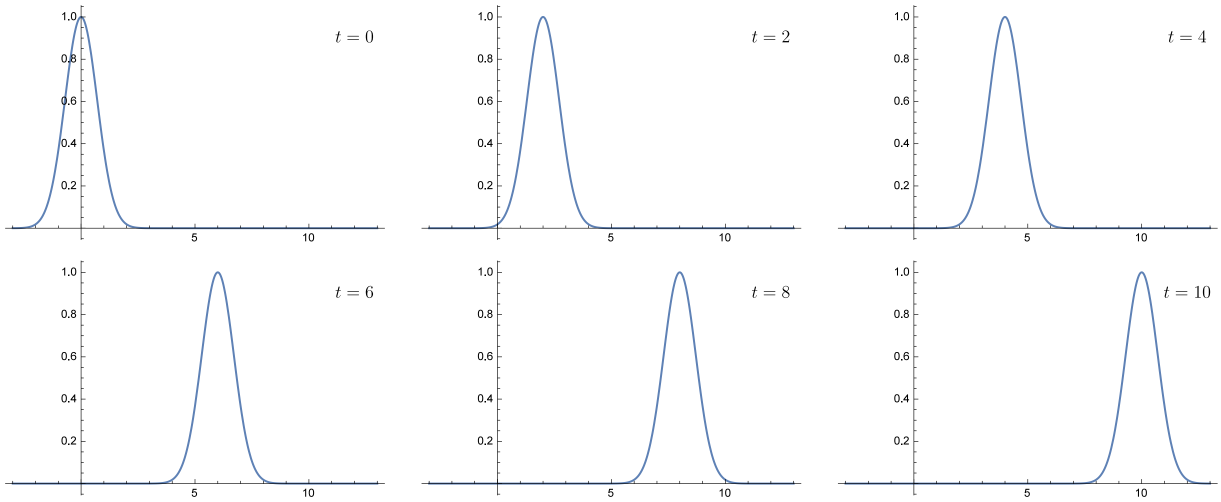

Figure 1: Traveling wave solution for different time moments.

Example 2.3 (Distributed source)**.** To practice the introduced approach, consider the initial value problem

$$\begin{aligned} u_t + cu_x &= f(t,x), \quad t > 0, x \in \mathbf{R}, \\ u(0,x) &= g(x), \quad x \in \mathbf{R}. \end{aligned} \tag{2.3}$$

The physical interpretation of problem (2.3) is that now we have a distributed source of the substance with the intensity (density per time unit) $f(t,x)$ at the time t and the position x.

Exercise 1. Deduce problem (2.3) from physical principles and the conservation law.

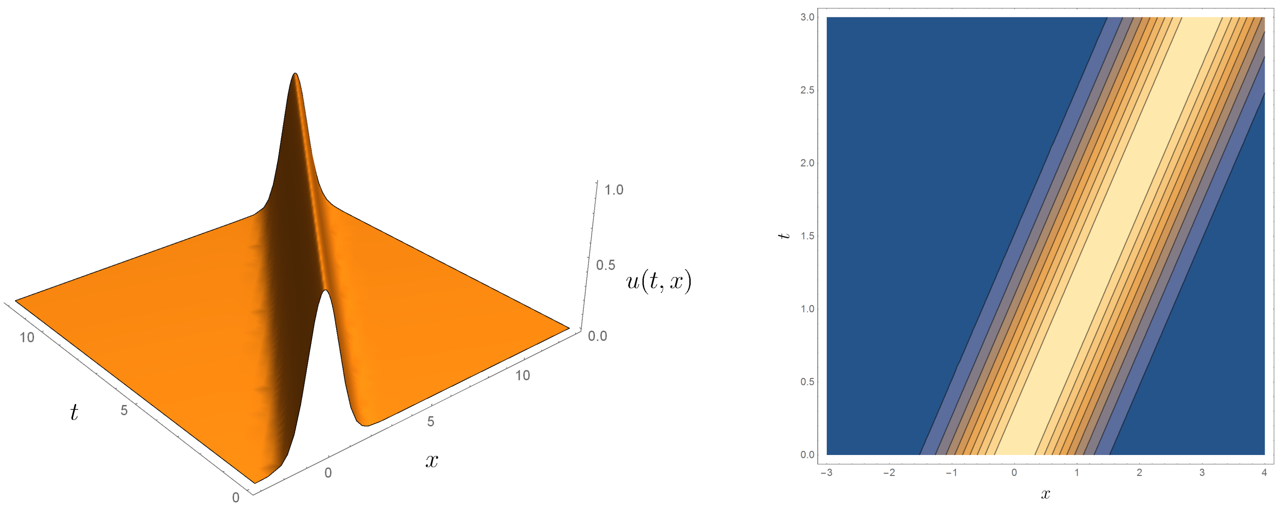

Figure 2: Traveling wave solution in the form of 3D plot (left) and contour plot (right).

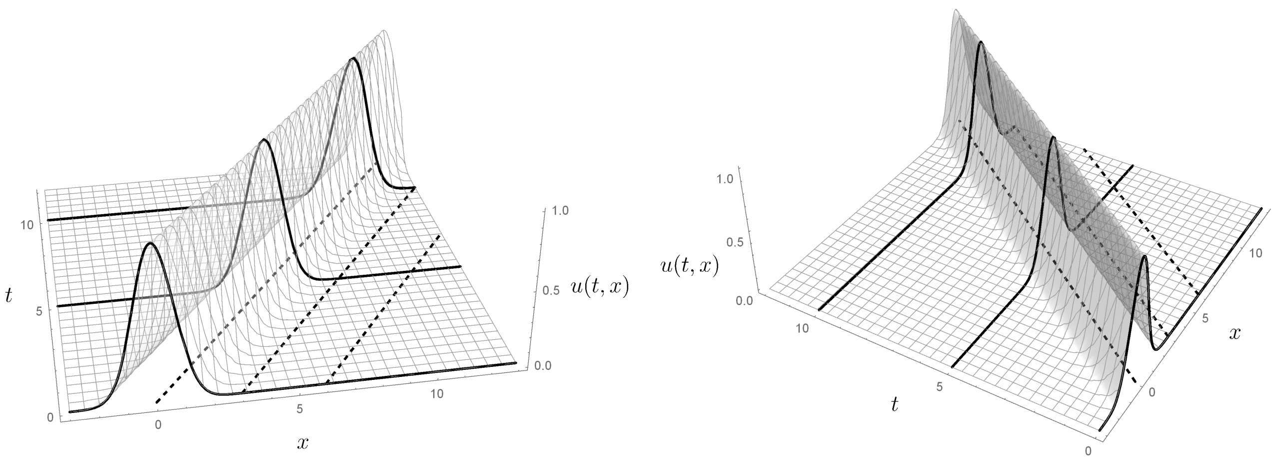

Figure 3: Traveling wave solution from two different viewpoints.

Using the same change of variables I find

$$v_\tau(\tau,\xi)=f(\tau,c\tau+\xi).$$

Integrating yields

$$v(\tau,\xi)=\int_0^\tau f(s,cs+\xi)\mathrm{d}s+F(\xi),$$

or, returning to the original variables,

$$u(t,x)=\int_0^t f(s,x-c(t-s))\mathrm{d}s+F(x-ct).$$

Using the initial condition I finally get

Theorem 2.4. *Let $g\in\mathcal{C}^{(1)}$ and $f,f_x\in\mathcal{C}$. Then the unique solution to* (2.3) *is given by*

$$u(t,x)=\int_0^t f\big(s,x-c(t-s)\big)\mathrm{d}s+g(x-ct).$$

Example 2.5 (Transport equation with decay)**.** Recall that one of the very basic ODE is the so called decay equation (you could have seen it with respect to, e.g., radioactive decay, it literally says that the rate of decay of some compound is proportional to the present mass)

$$u'=-au,$$

for some constant $a>0$. The solution to this equation is (by, e.g., the separation of variables)

$$u(t)=Ce^{-at}$$

and is used in, e.g., radioactive dating.

Consider now the PDE

$$u_t+cu_x=-u,$$

that is, physically, I assume that in addition to the transport process there is also decay proportional to the present density.

By the same change of the variables I get

$$v_\tau = -v,$$

and hence

$$v(\tau, \xi) = F(\xi)e^{-\tau},$$

where F is an arbitrary smooth function of ξ. Returning to the original variables,

$$u(t, x) = F(x - ct)e^{-t},$$

which results in

$$u(t, x) = g(x - ct)e^{-t}$$

for the initial condition

$$u(0, x) = g(x).$$

Now the solutions are not constant along the characteristics, but they are decaying along them representing a damped traveling wave.

Example 2.6 (A problem with a boundary condition)**.** Assume that now u denotes the concentration of some pollutant in a river and there is a source of this pollutant at the point $x = 0$ of the intensity $f(t)$ at time t. Mathematically it means that I consider only problem for $x > 0, t > 0$ for the equation

$$u_t + cu_x = 0$$

with $c > 0$.

Since it is natural to have the initial condition now only for $x > 0$ for some part on the first quadrant I will have to use the boundary condition. The characteristic $x = ct$ separates two regions, where I must use either the initial or boundary condition. If $x \geq ct$ then I can use the usual initial condition. If, however, $x < ct$ then for my general solution $u(t, x) = F(x - ct)$ I must have

$$F(-ct) = f(t),$$

or

$$F(\tau) = f(-\tau/c).$$

Finally I obtain a unique solution

$$u(t, x) = \begin{cases} g(x - ct), & x \geq ct, \\ f(t - x/c), & x \leq ct. \end{cases}$$

To guarantee that my solution is a classical one, I should also request that $f(0) = g(0)$ and $-cg'(0) = f'(0)$ hold.

Now assume that $c < 0$ in my problem. Therefore the characteristics will have a negative slope and each characteristic will cross both $x > 0$ and $t > 0$ half-lines. This will result in a well posed problem if and only if the initial and boundary values will be coordinated, otherwise, the problem will have no physical solution (see Fig. 4).

This is a first sign that in PDE a correct (physically) choice of initial and boundary conditions will result in a well posed problem, whereas an arbitrary assigning initial conditions can lead to no solutions at all.

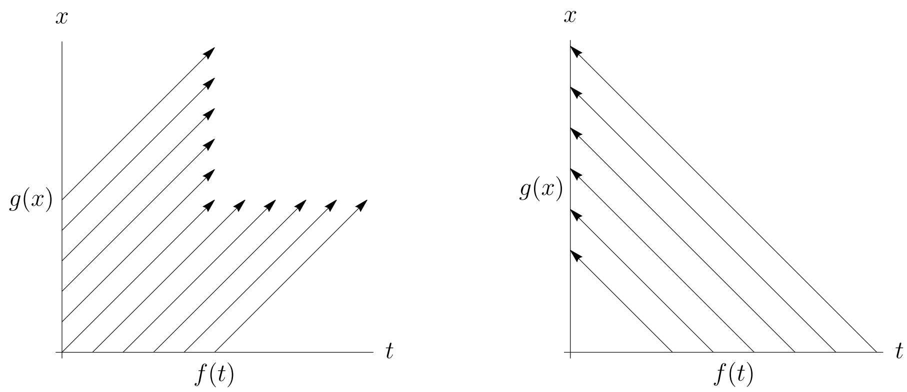

Figure 4: Correctly (left) and incorrectly (right) stated boundary conditions for a transport equation. The arrows indicate the direction of the transport with time, hence for the left panel I have $c > 0$ and for the right one $c < 0$.

Exercise 2. Solve linear k-dimensional transport equation

$$u_t + \boldsymbol{c} \cdot \nabla_{\boldsymbol{x}} u = 0,$$

where $\boldsymbol{c} = (c_1, \ldots, c_k)$, $\boldsymbol{x} = (x_1, \ldots, x_k)$, $\nabla_{\boldsymbol{x}} u = (u_{x_1}, \ldots, u_{x_k})$ is the spatial gradient of u.

2.3 Well posed problems in PDE theory

I already used the term "well posed" problems several times. It has, actually, a rigorous mathematical meaning. The following definition was originally coined by French mathematician Jacques Hadamard (1865—1963), who made a lot of contributions to the field of partial differential equations.

Definition 2.7. *The problem is called well possed is it has a solution, this solution is unique, and this solution depends continuously on the initial data and parameters.*

In the examples above I usually showed that a solution exists by presenting an explicit formula for this solution. The uniqueness was guaranteed by the fact that the signals are spread along the characteristics and the initial (and boundary) conditions uniquely prescribe values at one point of the characteristic. Here I will give a first example of continuous dependence on the initial data.

Proposition 2.8. *Consider a boundary-initial value problem for the transport equation:*

$$\begin{aligned} u_t + cu_x = 0, &\quad 0 < x < R,\, t > 0,\, c > 0 \\ u(0, x) = g(x), &\quad 0 < x < R, \\ u(t, 0) = f(t), &\quad t > 0. \end{aligned}$$

The solution to this problem depends continuously on the initial data.

Proof. First I multiply the equation by u and note

$$uu_t + cuu_x = \frac{1}{2}(u^2)'_t + \frac{c}{2}(u^2)'_x = 0.$$

Integrating in x from 0 to R yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^R u^2(t,x)\mathrm{d}x + c\Big(u^2(t,R) - u^2(t,0)\Big) = 0.$$

Hence, by positivity of c,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^R u^2(t,x)\mathrm{d}x \leq cf^2(t).$$

Now I integrate in t and use the condition $u(0,x) = g(x)$:

$$\int_0^R u^2(t,x)\mathrm{d}x \leq \int_0^R g^2(x)\mathrm{d}x + c\int_0^t f^2(s)\mathrm{d}s.$$

Finally I note that if u_1, u_2 are solutions to the problems with g_1, f_1 and g_2, f_2 respectively, then $u_1 - u_2$, due to linearity, solves the problem with $g_1 - g_2, f_1 - f_2$. Hence I end up with the estimate

$$\int_0^R \big(u_1(t,x) - u_2(t,x)\big)^2\mathrm{d}x \leq \int_0^R \big(g_1(x) - g_2(x)\big)^2\mathrm{d}x + c\int_0^t \big(f_1(s) - f_2(s)\big)^2\mathrm{d}s,$$

which shows that small change in the initial data yields small change in the solutions, which is essentially a rephrasing of the fact that the solution depends continuously on the initial data. ■

2.4 Test yourself

2.1. Formulate the condition for two vectors to be orthogonal (perpendicular).

2.2. What are the characteristics of $u_t + 2u_x = 0$?

2.3. What are the characteristics of $u_t + 2u_x = -u$?

2.4. Solve $u_t + 2u_x = 0$, $u(0,x) = x^3$.

2.5. Solve the equation $au_t + bu_x = 0$ with the initial condition $u(0,x) = g(x)$.

2.6. Solve $u_t + u_x = \sin t$ with $u(0,x) = \cos x$.

2.5 Solutions to the exercises

Exercise 1. Here I additionally assume that the substance I am following can be either created or destroyed inside (x_1, x_2) with the intensity f. That is, the total amount of substance created at time t is given by $\int_{x_1}^{x_2} f(t,s)\mathrm{d}s$. Therefore my conservation law now reads

$$u_t + q_x = f(t,x)$$

as required. ■

Exercise 2. Exactly as in one dimensional case I introduce new variables

$$(\tau, \boldsymbol{\xi}) = (t, \boldsymbol{x} - \boldsymbol{c}t),$$

where now $\boldsymbol{\xi}$, $\boldsymbol{x}$, $\boldsymbol{c}$ are vectors with the same number of coordinates. In these new coordinates the equation for $\boldsymbol{v}(\tau, \boldsymbol{\xi}) = u(t, \boldsymbol{x})$ becomes

$$v_\tau = 0,$$

therefore $v(t, \boldsymbol{\xi}) = F(\boldsymbol{\xi})$, or, in the original variables,

$$u(t, \boldsymbol{x}) = F(\boldsymbol{x} - \boldsymbol{c}t),$$

which is the answer for this exercise. ■